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1992 J. Phys. A: Math. Gen. 25 L713

(http://iopscience.iop.org/0305-4470/25/12/002)

J. Phys. A: Math. Gen. **25** (1992) L713–L717. Printed in the UK

LETTER TO THE EDITOR

'Non-classical' q-oscillator realization of the quantum SU(2) algebra

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Received 18 October 1991

Abstract. A new realization of $SU_q(2)$ algebra via two independent q-oscillators is found. In contrast to the 'classical' Jordan–Schwinger construction the proposed realization yields $SU_q(2)$ generators as linear functions of creation and annihilation q-bose operators. The functions of canonical basis $|j; m\rangle$ in q-oscillator representation are found. There are no 'classical' analogues of this realization—it 'disappears' if $q \to 1$.

The $SU_q(2)$ (or so-called 'quantum SU(2)') algebra is assumed to play an important role in problems of quantum field theory and statistical physics (for references see [1]). This algebra is formed by three generators J_0, J_+, J_-, obeying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm$$
$$[J_+, J_-] = (\sinh 2\omega J_0)/\sinh \omega. \tag{1}$$

In what follows we shall assume that $\omega > 0$.

The Casimir operator J^2 of $SU_q(2)$ has the expression

$$\hat{J}^2 = J_+J_- + (\cosh 2\omega(J_0 - 1/2))/2 \sinh^2 \omega. \tag{2}$$

The canonical basis of unitary finite-dimensional representation exists ψ_{jm} defined by the relations

$$J_0\psi_{jm} = m\psi_{jm}$$
$$J_-\psi_{jm} = \sigma_m\psi_{jm-1}$$
$$J_+\psi_{jm} = \sigma_{m+1}\psi_{jm+1}$$
$$\hat{J}^2\psi_{jm} = [(\cosh \omega(2j+1))/2 \sinh^2 \omega]\psi_{jm} \tag{3}$$

where (as for ordinary SU(2) algebra)) $|m| \leqslant j$, $2j+1 = 1, 2, \ldots$ is the dimension of the representation and

$$\sigma_m^2 = J^2 - (\cosh \omega(2m-1))/2 \sinh^2 \omega.$$

There is one more important 'quantum' algebra—so-called q-oscillator algebra. The latter is constructed from the number operator A_0 and q-bose creation–annihilation operators A_+, A_-. The commutation relations between these operators are

$$[A_0, A_\pm] = \pm A_\pm$$
$$[A_-, A_+] = \exp(-2\omega A_0) \tag{4}$$

(for $\omega = 0$ we arrive at the ordinary oscillator algebra). The q-oscillator algebra (4) (as well as its various modifications) has been considered (and rediscovered) in many papers [2-6].

The Casimir operator of q-oscillator algebra is

$$\hat{Q} = A_+ A_- + e^{-2\omega A_0}/(1 - e^{-2\omega}). \tag{5}$$

Writing the Casimir in the form

$$Q = e^{-2\omega\alpha}/(1 - e^{-2\omega}) \tag{6}$$

where α is arbitrary real parameter, one obtains the following canonical unitary representation in the basis $|n\rangle$

$$\begin{aligned} A_0|n\rangle &= (\alpha + n)|n\rangle \\ A_-|n\rangle &= \mu_n|n-1\rangle \\ A_+|n\rangle &= \mu_{n+1}|n+1\rangle \\ \hat{Q}|n\rangle &= Q(\alpha)|n\rangle \\ n &= 0, 1, 2, \ldots \end{aligned} \tag{7}$$

where

$$\mu_n^2 = e^{-2\omega\alpha}(1 - e^{-2\omega n})/(1 - e^{-2\omega}). \tag{8}$$

The representations (7), being infinite-dimensional, differ one from another by the value of the Casimir parameter α. For $\alpha = 0$ we have the q-bose algebra defined by the relation [2, 3, 6]

$$A_-^{(0)} A_+^{(0)} - q A_+^{(0)} A_-^{(0)} = 1. \tag{9}$$

For $\alpha \neq 0$ the corresponding relation is

$$A_-^{(\alpha)} A_+^{(\alpha)} - q A_+^{(\alpha)} A_-^{(\alpha)} = q^\alpha \tag{10}$$

where

$$q = \exp(-2\omega). \tag{11}$$

There are obvious relations between $A^{(0)}$ and $A^{(\alpha)}$

$$\begin{aligned} A_0^{(\alpha)} &= A_0^{(0)} + \alpha \\ A_\pm^{(\alpha)} &= \sqrt{q^\alpha}\, A_\pm^{(0)}. \end{aligned} \tag{12}$$

Because q-oscillator algebra seems to be simpler than $SU_q(2)$, it is natural to search for possible q-oscillator realizations of $SU_q(2)$. In [4, 5] the q-analogue of Jordan-Schwinger construction has been proposed. Let A_0, A_+, A_- and B_0, B_+, B_- be two independent (i.e. commuting) sets of q-oscillator operators both forming the representations with zero Casimir parameters $\alpha = \beta = 0$. Then the operators

$$\begin{aligned} J_0 &= (A_0 - B_0)/2 \\ J_- &= A_- B_+ \exp(\omega(A_0 + B_0 - 1)/2) \\ J_+ &= A_+ B_- \exp(\omega(A_0 + B_0 - 1)/2) \end{aligned} \tag{13}$$

form a representation of $SU_q(2)$. The dimension $2j+1$ of this representation is defined by the value j of the operator

$$\hat{j} = (A_0 + B_0)/2 \tag{14}$$

commuting with all generators J_0, J_-, J_+.

Formulae (13) yield the q-analogue of Jordan–Schwinger representation for $SU_q(2)$ [4, 5]. For $\omega = 0$ we obtain the well-known Jordan–Schwinger realization of angular momentum via two oscillators [7].

However there exists one more q-oscillator realization of $SU_q(2)$ having no classical analogue (i.e. it exists only for $\omega \neq 0$). Moreover, in contrast to (13), the new realization is linear on the creation-annihilation q-bose operators.

Let again $A_0, A_\pm, B_0, B_\pm$ be independent q-oscillator operators forming the representations of algebra (4) with Casimir parameters α and β. One can easy verify that operators

$$\begin{aligned} J_0 &= A_0 - B_0 \\ J_- &= (A_- \exp(\omega B_0) - B_+ \exp(\omega A_0))/\sqrt{2 \sinh \omega} \\ J_+ &= (A_+ \exp(\omega B_0) - B_- \exp(\omega A_0))/\sqrt{2 \sinh \omega} \end{aligned} \tag{15}$$

form the $SU_q(2)$ algebra with commutation relations (1). Note that the realization (15) does not explicitly depend on the Casimir parameters α and β. In the classical limit $(\omega \to 0)$ this realization 'disappears'.

Let us find the standard eigenstates ψ_{jm} (3) for the realization (15). For this it is sufficient to solve the two equations

$$J_0 \psi_{jm} = m \psi_{jm} \tag{16}$$

$$J^2 \psi_{jm} = \lambda_j \psi_{jm} \tag{17}$$

where

$$\lambda_j = \frac{\cosh \omega(2j+1)}{2 \sinh^2 \omega}. \tag{18}$$

Without loss of generality one can choose $m \geqslant 0$, so $j = m, m+1, \ldots$.

The functions ψ_{jm} may be represented in terms of q-oscillator eigenstates

$$\psi_{jm} = \sum_{n_A, n_B} W^{jm}_{n_A n_B} |n_A\rangle |n_B\rangle \tag{19}$$

where $|n_A\rangle, |n_B\rangle$ are eigenstates for A_0 and B_0:

$$A_0 |n_A\rangle = (\alpha + n_A)|n_A\rangle$$

$$B_0 |n_B\rangle = (\beta + n_B)|n_B\rangle.$$

From (16) one obtains the relation

$$n_A - n_B + \alpha - \beta = m. \tag{20}$$

Let us choose the Casimir parameters to be

$$\alpha - \beta = m. \tag{21}$$

So we obtain

$$n_A = n_B = 0, 1, 2 \ldots \tag{22}$$

and expansion (19) can be rewritten in the form

$$\psi_{jm} = \sum_{n=0}^{\infty} W_n^{jm}|n\rangle|n\rangle. \tag{23}$$

Substituting (23) into (17) one obtains the following recurrent relation for the coefficients W_n

$$a_{n+1}W_{n+1} + a_n W_{n-1} + b_n W_n = \lambda_j W_n \tag{24}$$

where

$$\begin{aligned} a_n &= (1 - e^{2\omega n})/4\sinh^2\omega \\ b_n &= e^{\omega(2n+1)}\cosh 2\omega m/2\sinh^2\omega. \end{aligned} \tag{25}$$

It is convenient to represent W_n^{jm} in the form

$$W_n^{jm} = W_0^{jm} P_n(\lambda_j;\, m). \tag{26}$$

For new functions P_n we have

$$P_0(\lambda_j;\, m) \equiv 1 \tag{27}$$

and

$$a_{n+1}P_{n+1} + a_n P_{n-1} + b_n P_n = \lambda_j P_n. \tag{28}$$

It is seen from (27), (28) (and from $a_0 = 0$) that functions P_n $(\lambda_j;\, m)$ are n-order polynomials of argument λ_j. These polynomials (together with their weight amplitude W_0) are uniquely determined by the three-term recurrent relation (28). To complete the calculations it is sufficient to note that coefficients a_n and b_n coincide with those defining the special class of Askey–Wilson polynomials [8]. Omitting the details of identification we represent the final result

$$(W_0^{jm})^2 = \exp(2\omega(m^2 - j^2))(1 - e^{-2\omega(2j+1)}) \tag{29}$$

$$P_n(\lambda_j;\, m) = q^{n(m+1/2)}\,{}_3\Phi_1\left(\begin{matrix} q^{-n}, q^{-x}, q^{x+2m+1} \\ q \end{matrix}\,\middle|\, q^{n+1-2m}\right) \tag{30}$$

where

$$q = e^{-2\omega}$$

$$x = j - m = 0, 1, 2, \ldots.$$

${}_3\Phi_1$ is the so-called basic hypergeometric function defined by [8]

$${}_3\Phi_1\left(\begin{matrix} a, b, c \\ d \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k(k+1)/2}(a)_k(b)_k(c)_k z^k}{(q)_k(d)_k} \tag{31}$$

where $(a)_k = (1-a)(1-qa)\ldots(1-a\cdot q^{k-1})$ is the Pochhammer q-symbol.

Formulae (29), (30) completely determine the coefficients W_n^{jm} in expansion (23). So we have obtained the function ψ_{jm} in the q-oscillator representation.

Note that for the lowest state $j = m$ one obtains the simple formula ('Planck distribution')

$$\psi_{jj} = \sqrt{1 - q^{2j+1}} \sum_{n=0}^{\infty} q^{n(j+1/2)}|n\rangle|n\rangle. \tag{32}$$

It is again seen from (32) that this distribution exists only for $q \neq 1$ (formally speaking, the case $q \to 1$ corresponds to an 'infinitely increasing temperature').

Note, finally, that analogous representations exist also for the $SU_q(1,1)$ algebra. Indeed, the operators

$$
\begin{aligned}
N_0 &= A_0 - B_0 \\
N_- &= (A_- \exp(-\omega B_0) - B_+ \exp(-\omega A_0))/\sqrt{2 \sinh \omega} \\
N_+ &= (A_+ \exp(-\omega B_0) - B_- \exp(-\omega A_0))/\sqrt{2 \sinh \omega}
\end{aligned}
\tag{33}
$$

form a $SU_q(1,1)$ algebra with commutation relations

$$
\begin{aligned}
[N_0, N_\pm] &= \pm N_\pm \\
[N_-, N_+] &= (\sinh 2\omega N_0)/\sinh \omega.
\end{aligned}
\tag{34}
$$

In contrast to (15) the q-bose operators in (33) obey the relations with inverted sign of ω:

$$
\begin{aligned}
[A_-, A_+] &= \exp(\omega A_0) \\
[B_-, B_+] &= \exp(\omega B_0) \qquad \omega > 0.
\end{aligned}
\tag{35}
$$

The representations of both discrete and continuous series of $SU_q(1,1)$ can be obtained from (33) in a similar way.

The author is grateful to Professor Ya Granovskii for helpful discussions.

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